The exotic Galilei group and the "Peierls substitution"

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February 1, 2008

Abstract

Taking advantage of the two-parameter central extension of the planar Galilei group, we construct a non relativistic particle model in the plane. Owing to the extra structure, the coordinates do not commute. Our model can be viewed as the non-relativistic counterpart of the relativistic anyon considered before by Jackiw and Nair. For a particle moving in a magnetic field perpendicular to the plane, the two parameters combine with the magnetic field to provide an effective mass. For vanishing effective mass the phase space admits a two-dimensional reduction, which represents the condensation to collective "Hall" motions, and justifies the rule called "Peierls substitution". Quantization yields the wave functions proposed by Laughlin to describe the Fractional Quantum Hall Effect.

1 Introduction

The rule called "Peierls substitution" [1] says that a charged particle in the plane subject to a strong magnetic field B and to a weak electric potential $V(x, y)$ will stay in the lowest Landau level, so that its energy is approximately $E = eB/(2m) + \epsilon$, where ϵ is an eigenvalue of the potential $eV(X, Y)$ alone. The operators X and Y satisfy, however, the anomalous commutation relation

$$[X, Y] = \frac{i}{eB}. \tag{1}$$

Similar ideas emerged, more recently, in the context of the Fractional Quantum Hall Effect [2], where it is argued [3] that *the system condensates into a collective ground state*. This "new state of matter" is furthermore represented by the "Laughlin" wave functions (22) below, which all belong to the lowest Landau level [4].

Dunne, Jackiw, and Trugenberger [5] justify the Peierls rule by considering the $m \to 0$ limit, reducing the classical phase space from four to two dimensions, parametrized by non-commuting coordinates X and Y, whereas the potential $V(X, Y)$ becomes an effective Hamiltonian. While this yields the essential features of the Peierls substitution, it has the disadvantage that the divergent ground state energy $eB/(2m)$ has to be removed by hand. In this Letter, we derive a similar model from *first principles*, without resorting to such an unphysical limit.

First we construct, following Souriau [6], a model for a non-relativistic particle in the plane associated with the two-parameter central extension [7, 8, 9, 10] of the Galilei group. Our model, parametrized by the mass, m, and a new invariant, κ, turns out to be the non-relativistic limit of the relativistic anyon model of Jackiw and Nair [11].

For a free particle the usual equations of motions hold unchanged and κ only contributes to the conserved quantities, (6). More importantly, it yields non-commuting position coordinates, see below. Minimal coupling to an external gauge field unveils, however, new and interesting phenomena, which seem to have escaped attention so far. The interplay between the internal structure associated with κ and the external magnetic field B yields, in fact, an effective mass m^*. For vanishing effective (rather than real) mass, we get some curiously simple motions, which satisfy a kind of *generalized Hall law*, Eq. (11) below. For a constant electric field the usual cycloidal motions degenerate to a pure drift of their guiding centers alone. Such motions form a two-dimensional submanifold of the four-dimensional space of motions. Reduction to this subspace is the classical manifestation of Laughlin's condensation into a collective motion. Then the quantization of the reduced model allows us to recover the Laughlin description.

2 Exotic particle in the plane

First we construct a classical model of our "exotic" particle in the plane. Let us start with the Faddeev-Jackiw framework [12]. A mechanical system is described by the classical action $\int\theta$ defined through the "Lagrange one-form" $\theta = a_\alpha d\xi^\alpha - Hdt$, where $\xi = (\vec{r}, \vec{v})$ is a point in phase space. The Euler-Lagrange equation is expressed using $\omega = \frac{1}{2}\omega_{\alpha\beta}d\xi^\alpha \wedge d\xi^\beta$, the $t = \text{const}$ restriction of the two-form $d\theta$, as

$$\omega_{\alpha\beta}\dot{\xi}^\beta = \partial_{\xi^\alpha} H. \tag{2}$$

For a system with a first-order Lagrangian $\mathcal{L} = \mathcal{L}(\vec{x}, \vec{v}, t)$, for example, one can chose in particular $\theta = \mathcal{L}\,dt$; when ω is regular, we get Hamilton's equations. The construction works, however, under more general conditions: on the one hand, not all one-forms θ come from a Lagrangian $\mathcal{L}$ which would only depend on position, velocity and time [13]. On the other hand, the two-form ω can suffer singularities, necessitating "Hamiltonian reduction", which amounts to eliminating some of variables and writing the reduced one-form using intrinsic canonical coordinates on the reduced manifold [12].

The Faddeev-Jackiw framework is actually equivalent to that of Souriau [6], who proposed to describe the dynamics by a closed two-form, σ, of constant rank on the "evolution space" $\mathcal{V}$ of positions $\vec{r}$, velocities $\vec{v}$, and time t. Then the classical motions are the integral curves of the null space of σ, viz

$$(\dot{\vec{r}}, \dot{\vec{v}}, \dot{t}) \in \ker\sigma. \tag{3}$$

Writing σ as $\omega - dH \wedge dt$, the Euler-Lagrange equations (2) are recovered. Being closed, σ is furthermore locally $d\theta$, showing that the two approaches are indeed equivalent.

Working with the two-form σ is actually more convenient as working with the one-form θ. For example, a symmetry is a transformation which leaves σ invariant, while the Lagrange one-form θ changes by a total derivative.

Souriau [6] actually goes one step farther, and (as advocated also by Crnkovic and Witten [15]), argues that the fundamental space to look at is $\mathcal{M}$, the *space of solutions of the equations of motion*. Souriau calls this abstract substitute of the phase space the *space of motions*. In our case, $\mathcal{M}$ is the simply the set of motion curves in the evolution space $\mathcal{V}$.

Our classical particle model is then constructed as follows. Let us recall that the elementary particles correspond to irreducible, unitary representations of their symmetry groups. According to geometric quantization, though, these representations are associated with some coadjoint orbits of the symmetry group [6, 14]; the idea of Souriau [6] was to view these orbits, endowed with their canonical two-forms, as *spaces of motions*.

Now, as discovered by Lévy-Leblond [7], the planar Galilei group admits a two-parameter central extension, parametrized by two real constants m and κ (see, e.g., [16]). The new invariant κ has the dimension of $\hbar/c^2$. The coadjoint orbits of the doubly-extended Galilei group coincide with those of the singly-extended one, but carry a modified symplectic structure. The interesting ones are those associated with the mass $m > 0$ and $\kappa \neq 0$; they are $\mathcal{M} = \mathbf{R}^4$ with coordinates (v_i) and (q^i), endowed with the noncanonical *twisted-in-the-wrong-way* symplectic two-form

$$\omega = m\ dv_i \wedge dq^i + \tfrac{1}{2}\kappa\,\varepsilon_{ij}\,dv^i \wedge dv^j. \tag{4}$$

Owing to the new term in (4), the Poisson bracket of the configuration coordinates is nonvanishing, $\{x, y\} = \kappa/m^2$. For these orbits, the evolution space is $\mathcal{V} = \mathcal{M} \times \mathbf{R} \simeq \mathbf{R}^5$, endowed with the two-form

$$\sigma = m\ dv_i \wedge (dr^i - v^i dt) + \tfrac{1}{2}\kappa\,\varepsilon_{ij}\,dv^i \wedge dv^j. \tag{5}$$

This two-forms is exact, namely $\sigma = d\theta$ with $\theta = mv_i dr^i - \frac{1}{2}m|\vec{v}|^2 dt + \frac{1}{2}\kappa\epsilon_{ij}v^i dv^j$. However, because of the "exotic" contribution, it is *not* of the form $\mathcal{L}\,dt$ with a first-order Lagrangian $\mathcal{L}$ [13]; thus, this model has *no ordinary Lagrangian*. Both generalized formalisms work nevertheless perfectly, and we choose to pursue along these lines. (Let us mention that a Lagrangian could be constructed—but it would be acceleration-dependent [10].)

Most interestingly, the "exotic" term $\frac{1}{2}\kappa\,\varepsilon_{ij}\,dv^i \wedge dv^j$ in (4) has already been used, namely to describe relativistic anyons [11]; our presymplectic form (5) appears to be the non relativistic limit of that in Ref. [11] when their spin, s, is identified with our parameter κ. (We believe in fact that our particles are indeed non-relativistic anyons.) Group contraction of the (trivially) centrally-extended Poincaré group yields furthermore the doubly-extended planar Galilei group [9].

It is readily seen that the modified two-form (5) yields the usual equations of free motions, despite the presence of the new invariant κ. The two-form (5) on $\mathcal{V}$ flows down to $\mathcal{M}$ as ω in (4) along the projection $(\vec{r}, \vec{v}, t) \to (\vec{q}, \vec{v})$, where $\vec{q} = \vec{r} - \vec{v}t$. The space of free motions is hence $\mathcal{M}$, endowed with the symplectic form ω.

For completeness, let us mention that σ is invariant with respect to the natural action of the Galilei group on $\mathcal{V}$ whose "moment map" [6] consists of the conserved quantities

$$\left\{\begin{array}{ll} \jmath & = m\,\vec{r} \times \vec{v} + \frac{1}{2}\kappa|\vec{v}|^2, \\[10pt] k_i & = m(r_i - v_i t) + \kappa\,\varepsilon_{ij}v^j, \\[10pt] p_i & = mv_i, \\[10pt] h & = \frac{1}{2}m|\vec{v}|^2. \end{array}\right. \tag{6}$$

These same quantities were found before (see [8, 9]), using rather different methods. Let us observe that, owing to the exotic structure, the angular momentum $\jmath$ and the boosts $\vec{k}$ in (6) contain new terms (which are, however, also separately conserved). By construction, they satisfy the commutation relations of the doubly-extended planar Galilei group which only differ from the usual ones in that the boosts no longer commute, $\{k_i, k_j\} = \kappa\varepsilon_{ij}$, cf. [7, 8, 9].

Let us now put our charged particle into an external electromagnetic field $F = (\vec{E}, B)$. Applying as in [6] the minimal coupling prescription $\sigma \to \sigma + eF$, the system is now described by the two-form

$$\sigma = (m\,dv_i - eE_i dt) \wedge (dr^i - v^i dt) + \tfrac{1}{2}\kappa\,\varepsilon_{ij}\,dv^i \wedge dv^j + \tfrac{1}{2}eB\,\varepsilon_{ij}\,dr^i \wedge dr^j \qquad (7)$$

on the evolution space $\mathcal{V}$. It is interesting to note that our two-form (7)—which is again exact if F is exact, but is in no way Lagrangian—is the non-relativistic limit of the relativistic expression in Ref. [17]. A short computation shows that a tangent vector $(\delta\vec{r}, \delta\vec{v}, \delta t)$ satisfies the Euler-Lagrange equations (3) when

$$\begin{cases} m^*\delta r^i = m\left(v^i - \dfrac{e\kappa}{m^2}\,\varepsilon^i_j E^j\right)\delta t, \\[2ex] m\,\delta v^i = e\left(E^i\delta t + B\,\varepsilon^i_j\delta r^j\right), \\[2ex] m\,v_i\delta v^i = e\,E_i\delta r^i, \end{cases} \qquad \text{where} \qquad m^* = m - \frac{\kappa e B}{m}. \qquad (8)$$

If the effective mass m^* is nonzero, the third equation is automatically satisfied; the middle one becomes

$$m^*\delta v_i = e\left(E^i + B\,\varepsilon^i_j v^j\right)\delta t. \qquad (9)$$

Thus, for $\kappa \neq 0$, the velocity $\delta\vec{r}/\delta t$ and the "momentum" $\vec{v}$ are *different* (not even parallel); it is the latter which satisfies the Lorentz equations of motion (9) with effective mass m^*.

If, however, *the effective mass m^* vanishes*, i.e., when the magnetic field B takes the critical (constant) value

$$B = \frac{m^2}{e\kappa}, \qquad (10)$$

then σ suffers singularities. The curious "motions" with instantaneous propagation can be avoided and we can still have consistent equations of motion, *provided* $v^i = (e\kappa/m^2)\varepsilon^i_j E^j$. But this latter condition, together with Eq. (10), astonishingly reads

$$v^i = \frac{1}{B}\,\varepsilon^i_j E^j. \qquad (11)$$

This generalized *Hall law* requires that particles move with "momentum" $\vec{v}$ perpendicular to the electric field and determined by the ratio of the (possibly position and time dependent) electric and the (constant) magnetic fields.

Assume, from now on that, the electric field $\vec{E} = -\vec{\nabla} V$ be time-independent. On the three-dimensional submanifold $\mathcal{W}$ of $\mathcal{V}$ defined by Eq. (11), the two-form (7) induces a well-behaved closed two-form $\sigma_{\mathcal{W}}$ of rank 2. Upon defining the new "position" variables

$$Q^i = r^i - \frac{mE^i}{eB^2}, \tag{12}$$

one readily finds that

$$\sigma_{\mathcal{W}} = \tfrac{1}{2} eB\, \varepsilon_{ij}\, dQ^i \wedge dQ^j - dH \wedge dt \tag{13}$$

with the (reduced) Hamiltonian $H = eV(\vec{r}) + m|\vec{E}|^2/(2B^2)$. The second term, here, represents the drift energy. The equations of motion are simply

$$\begin{cases} \dot{Q}^i = \dfrac{1}{B}\, \varepsilon^i_j E^j, \\ \dot{H} = 0, \end{cases} \tag{14}$$

confirming that the Hamiltonian descends to the reduced space of motions. The latter is two-dimensional and endowed with a symplectic two-form, we call Ω, inherited from $\sigma_{\mathcal{W}}$. Easy calculation shows that $\partial H/\partial Q^i = -eE_i$, hence

$$H = eV(X, Y) \tag{15}$$

where (X, Y) are coordinates on the reduced space of motions, $\mathcal{H}$, obtained by integrating the equations of motion (cf. Eq. (17) below). Note that the drift energy has been absorbed into H by the redefinition of the position, Eq. (12). At last, one finds that the coordinates X and Y on $\mathcal{H}$ have anomalous Poisson bracket

$$\{X, Y\} = \frac{1}{eB}. \tag{16}$$

In conclusion, we have established via Eqs (15) and (16) the classical counterpart of the Peierls rule. Let us insist that our construction does not rely on any unphysical limit of the type $m \to 0$, rather it uses the new freedom of having a vanishing effective mass.

3 Hall motions

Let us assume henceforth that the electric field $\vec{E}$ is constant. The equations of motion are readily solved. For nonzero effective mass m^*, i.e., when the magnetic field does not take the critical value (10), one recovers the usual motion, composed of uniform rotation (but with modified frequency eB/m^*) plus the drift of the guiding center.

When the magnetic field takes the critical value (10) and when the constraint (11) is also satisfied, velocity and "momentum" become the same, $\vec{v} = \delta\vec{r}/\delta t$, so that the constraint (11) requires that all particles move collectively, according to ... *Hall's law* !

This is understood by noting that for vanishing effective mass $m^* = 0$, the circular motion degenerates to a point, and we are left with the uniform drift of the guiding center alone.

The reduced space of motions $\mathcal{H}$ (we suggestively called the *space of Hall motions*) can now be described explicitly. It is parametrized (see Eqs (12) and (14)) by the coordinates $(X, Y) \equiv (R^i)$ where

$$R^i = Q^i - \frac{1}{B}\,\varepsilon^i_j E^j\, t. \tag{17}$$

The constraint (11) implies now that $\delta\vec{v} = 0$; the induced presymplectic two-form on the three-dimensional submanifold $\mathcal{W}$ is hence simply eF. The symplectic structure of the space of Hall motions is therefore

$$\Omega = \tfrac{1}{2} eB\varepsilon_{ij}\, dR^i \wedge dR^j = eB\, dX \wedge dY. \tag{18}$$

The coordinates X and Y have therefore the Poisson bracket (16).

The symmetries and conserved quantities can now be found. Firstly, the ordinary space translations $(\vec{r}, \vec{v}, t) \to (\vec{r} + \vec{c}, \vec{v}, t)$ are symmetries for the reduced dynamics, since they act on $\mathcal{H}$ according to $\vec{R} \to \vec{R} + \vec{c}$. The associated conserved quantities identified as the "reduced momenta" are linear in the position and time; they read

$$P_i = -eB\varepsilon_{ij} R^j = -eB\varepsilon_{ij} Q^j - eE_i\, t. \tag{19}$$

(Their conservation can also be checked directly using the Hall law (11)). The reduced momenta do not commute but have rather the Poisson bracket of "magnetic translations",

$$\{P_X, P_Y\} = eB. \tag{20}$$

The time translations $(\vec{r}, \vec{v}, t) \to (\vec{r}, \vec{v}, t+\tau)$ act on $\mathcal{H}$ according to $R^i \to R^i - \varepsilon^i_j E^j \tau / B$, which is a combination of space translations. The reduced Hamiltonian is (see (15))

$$H = -e\vec{E}\cdot\vec{R} = -e\vec{E}\cdot\vec{r} \tag{21}$$

and is related to the reduced momenta by $H = -\vec{E} \times \vec{P}/B$. The remaining Galilean generators $\jmath$ and $\vec{k}$ are plainly broken by the external fields. (The system admits instead "hidden" symmetries that will be discussed elsewhere.)

It is amusing to compare the reduced expressions with the conserved quantities $\vec{p}$ and h associated with these same symmetries acting on the original (but "exotic") evolution space $\mathcal{V}$ "before" reduction. We find $\vec{p} = m\vec{v} + \vec{P}$ and $h = \frac{1}{2}m|\vec{v}|^2 + eV \equiv \frac{1}{2}m|\vec{v}|^2 + H$, where the velocity is of course fixed by the Hall law. Our reduced expressions are hence formally obtained by the "$m \to 0$ limit", as advocated in Ref. [5].

Our construction here appears as a nice illustration of Hamiltonian reduction [12]. The restriction to the $t = \text{const}$ phase space of our two-form σ in (7) is a closed two-form, ω. The generic case, $m^* \neq 0$, above arises when ω is regular, so that the matrix ω is invertible. On the other hand, vanishing effective mass, $m^* = 0$, as in (10), means precisely that ω is singular. In Faddeev-Jackiw language, our reduction amounts to eliminating the velocities by the constraint (11) to yield X and Y as conjugate canonical variables and H as the Hamiltonian, on reduced space. This is seen by writing $\sigma_{\mathcal{W}}$, in (13), as $d\theta_{\mathcal{W}}$ with Lagrange form $\theta_{\mathcal{W}} = \frac{1}{2}eB\,\varepsilon_{ij}\,Q^i dQ^j - H(\vec{v}, \vec{Q})dt$; note that the dv^i are absent and the $\vec{v}$ only appear in the Hamiltonian and are determined by (11).

4 Quantization of the Hall motions

The quantization is simplified by observing that the space of Hall motions is actually the same of that of a one-dimensional harmonic oscillator with cyclotron frequency eB/m. The standard procedures [6, 14] can therefore be applied.

Let us assume that we work on the entire plane and introduce the complex coordinate $Z = \sqrt{eB}(X + iY)$; the symplectic form (18) is then $\Omega = d\bar{Z} \wedge dZ/(2i)$, hence $\{\bar{Z}, Z\} = 2i$. Now Ω is exact, $\Omega = d\Theta$ with the choice $\Theta = (\bar{Z}dZ - Zd\bar{Z})/(4i)$ corresponding to the "symmetric gauge". The prequantum line-bundle is therefore trivial; it carries a connection with covariant derivative $D = \partial - \frac{1}{4}\bar{Z}$ along ∂. Choosing the antiholomorphic polarization, spanned by $\bar{\partial}$, yields the wave "functions" as half-forms $\psi(Z, \bar{Z})\sqrt{dZ}$ that are covariantly constant along the polarization, i.e., such that $\bar{D}\psi = 0$. This yields

$$\psi(Z, \bar{Z}) = f(Z)e^{-|Z|^2/4} \tag{22}$$

with $f(Z)$ holomorphic, $\bar{\partial}f = 0$. The the inner product is $\langle f, g\rangle = \int_{\mathcal{H}} \overline{f(Z)}g(Z)e^{-|Z|^2/2}\,\Omega$. We recover hence the "Bargmann-Fock" [18] wave functions proposed by Laughlin [3], and by Girvin and Jach [2] to explain the FQHE. These wave functions span a subspace of the Hilbert space of the "unreduced" system and, indeed, represent the ground states in the FQHE [4]. (The details of the projection to the lowest Landau level are not yet completely clear, though [19].)

The quantum operator associated to the polarization-preserving classical observables are readily found [14]. For example, the quantum operators $\hat{Z}$ and $\bar{Z}$ are given as $\hat{Z}\psi = (-2\bar{D} + Z)\psi$ and $\hat{\bar{Z}}\psi = (2D + \bar{Z})\psi$. Acting on the holomorphic part alone, this yields for the complex momenta $\widehat{P} = \widehat{Z}$ and $\widehat{\bar{P}} = \widehat{\bar{Z}}$

$$\begin{cases} [\widehat{Z}f](Z) = Zf(Z), \\[2ex] [\widehat{\bar{Z}}f](Z) = 2\,\partial f(Z). \end{cases} \tag{23}$$

(See also [2].) Quantization of polarization-preserving observables takes Poisson brackets into commutators; in particular, we have $\frac{1}{2}[\widehat{\bar{Z}}, \widehat{Z}\,] = 1$, so that $\widehat{Z}$, $\widehat{\bar{Z}}$ and the identity span the Heisenberg algebra, just like their classical counterparts.

Being a combination of translations, the reduced Hamiltonian (21)—different from the usual quadratic oscillator Hamiltonian—becomes $H = (\bar{\mathcal{E}}Z + \mathcal{E}\bar{Z})/(2B)$, once we have put $\mathcal{E} = \sqrt{eB}(E_1 + iE_2)$. Its quantum counterpart is found as $\hat{H} = (\bar{\mathcal{E}}\hat{Z} + \mathcal{E}\hat{\bar{Z}})/(2B)$ with (23). For an electric field in the x direction, for example,

$$[\widehat{H}f](Z) = a(2\partial + Z)f(Z), \tag{24}$$

where $a = \frac{1}{2}E\sqrt{e/B}$. (The subtle problem of ordering does not arise here.) The eigenfunctions of $\hat{H}$ in (24) are readily found as $f(Z) = Ae^{-(Z-Z_0)^2/4}$ associated with the (real) eigenvalue $\epsilon = aZ_0$, cf. [4].

Thus the Peierls rule is confirmed also at the quantum level, for a linear potential.

5 Discussion

In the spirit of Dirac, we believe that "*it would be surprising if Nature would not seize the opportunity to use the new invariant* κ." While it plays little rôle as long as the particle is free, this invariant becomes important when the particle is coupled to an external field: albeit the classical motions are similar to those in the case $\kappa = 0$, it yields effective terms responsible for the reduction we found here. This curious interplay between the "exotic" structure and the external magnetic field is linked to the two-dimensionality of space and to the Galilean invariance of the theory. Mathematically, the second extension parameter arises owing to the commutativity of planar rotations—just like for exotic statistics of anyons [20]. The physical origin of κ is, perhaps, the band structure. In a solid the effective mass can be as much as 30 times smaller than that of a free electron. Our formula (10) could indeed serve to measure the new invariant κ using the data in the FQHE.

Had we worked over the two-torus $\mathbf{T}^2$ rather than over the whole plane, prequantization would require the integrality condition $\int \Omega = 2\pi\hbar N$ for some integer N [6, 14]. The actual meaning of this condition is that the "Feynman" factor

$$\exp\left(\frac{i}{\hbar}\int\theta\right) \tag{25}$$

be well-defined, independently of the choice of the one-form θ [21].

Representing $\mathbf{T}^2$ by a rectangle of sides L_x and L_y would then imply the well-known magnetic flux quantization condition $eBL_xL_y = 2\pi\hbar N$[4], analogous to the Dirac quantization of monopoles. Furthermore, the non-simply-connectedness of the torus implies that the factor (25) can have different inequivalent meanings, labeled by the characters of $\mathbf{Z} \times \mathbf{Z}$, the homotopy group of the two-torus [6, 21, 22].

6 Acknowledgement

We are indebted to Prof. R. Jackiw for acquainting us with the Peierls substitution and for advice, and to Profs. J. Balog, P. Forgács and Z. Horváth for discussions. Correspondence with Profs. V. P. Nair and G. Dunne is also acknowledged.

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